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ON GAUSS MAPS IN POSITIVE CHARACTERISTICS

ATSUSHI ITO

Abstract. In [FI2], K. Furukawa and the author study Gauss maps in positive characteristics. In this note, we review [FI2].

1. Introduction

Let $X \subset \mathbb{P}^N$ be an n-dimensional projective variety over an algebraically closed field of characteristic $p \geq 0$. Then we can define the *Gauss map* γ_X of X to be the rational map

$$\gamma_X : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$$

which maps a smooth point $x \in X$ to the embedded tangent space $\mathbb{T}_x X \subset \mathbb{P}^N$. Here $\mathbb{G}(n, \mathbb{P}^N)$ is the Grassmannian which parametrizes n-dimensional linear subvarieties of $\mathbb{P}^N$.

Example 1.1. (1) If $X \subset \mathbb{P}^N$ is an n-dimensional linear subvariety, $\mathbb{T}_x X = X$ holds for any $x \in X$, i.e. γ_X is a constant map. In fact, γ_X is a constant map for a variety $X \subset \mathbb{P}^N$ if and only if X is linear.
(2) Let $X \subset \mathbb{P}^N$ be the projective cone of $Y \subset \mathbb{P}^{N-1}$ with the vertex $v \in \mathbb{P}^N$. Then γ_X is constant on the line $\overline{vy}$ for each $y \in Y$.
(3) If $C \subset \mathbb{P}^N$ is a nonlinear curve in $p = 0$, it is known that γ_C is birational onto the image. Geometrically, this means that a general tangent line of C is tangent to C at only one point.

If we take a local parametrization of X, we can compute γ_X explicitly as follows.

Example 1.2. Let $X := \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ be the twisted cubic curve, which is locally parametrized as $[1 : t] \mapsto [1 : t : t^2 : t^3] \in \mathbb{P}^3$. For $x = [1 : t]$, the embedded tangent space $\mathbb{T}_x X \subset \mathbb{P}^3$ is the line spanned by $[1 : t : t^2 : t^3]$ and $[0 : 1 : 2t : 3t^2]$, where the latter point is obtained by differentiating $1, t, t^2, t^3$ by t. Then we have

$$\begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -t^2 & -2t^3 \\ 0 & 1 & 2t & 3t^2 \end{bmatrix},$$

where $\sim$ means that the farmer matrix is transformed to the latter one by elementary transformations of raws. Hence the Gauss map $\gamma_X : X \to$

$\mathbb{G}(1, \mathbb{P}^3)$ is locally described as

$$[1 : t] \mapsto \begin{bmatrix} -t^2 & -2t^3 \\ 2t & 3t^2 \end{bmatrix} \in \mathbb{A}^4 \subset \mathbb{G}(1, \mathbb{P}^3).$$

By this description, γ_X is an embedding if $p \neq 2$, but is not birational if $p = 2$.

Example 1.3. Assume $p > 0$ and let c be a positive integer prime to p. Let $X := \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, which is parametrized as $[1 : t] \mapsto [1 : t : t^{cp} : t^{cp+1}]$. Then $\gamma_X : X \to \mathbb{G}(1, \mathbb{P}^3)$ is locally described as

$$[1 : t] \mapsto \begin{bmatrix} t^{cp} & 0 \\ 0 & t^{cp} \end{bmatrix} \in \mathbb{A}^4 \subset \mathbb{G}(1, \mathbb{P}^3).$$

Thus a general fiber of γ_X is (non-reduced) c points. Geometrically, this means that a general tangent line of X is tangent to X at c points.

Remark 1.4. In [FI1], Furukawa and the author study Gauss maps of toric varieties.

About general fibers of Gauss maps, the following are known: If $p = 0$, (the closure of) a general fiber of γ_X is a linear subvariety for any X (see P. Griffiths and J. Harris [GH, (2.10)], F.L. Zak [Za, I, 2.3. Theorem (c)]). We will omit the words "the closure of" sometimes for simplicity. In particular, X is birational to $\gamma_X(X) \times \mathbb{A}^\delta$ for $\delta = \dim X - \dim \gamma_X(X)$, and the field extension $K(X)/K(\gamma_X(X))$ between function fields is purely transcendental. In $p > 0$, Furukawa [Fur] show that the same statement holds if γ_X is separable.

On the other hand, a general fiber of Gauss maps can be a union of points as we see in Example 1.3 (such examples were first found by H. Kaji [Ka1, Example 4.1] and J. Rathmann [Ra, Example 2.13]), and can be a nonlinear variety (S. Fukasawa [Fuk1, Section 7]) in $p > 0$. More generally, Fukasawa show that any projective variety appears as a general fiber of γ_X for some X:

Theorem 1.5 ([Fuk2])**.** *Assume $p > 0$. Let $F \subset \mathbb{P}^{N'}$ be a projective variety. If $n \geq N'$, there exists an n-dimensional projective variety $X \subset \mathbb{P}^N$ such that a general fiber of γ_X (with the reduced structure) is projectively equivalent to $F \subset \mathbb{P}^{N'}$.*

The following is an example of γ_X whose general fiber is a conic.

Example 1.6. Let $\varphi : (k^\times)^2 \hookrightarrow \mathbb{P}^3$ be an embedding defined by $(s, t) \mapsto [1 : s : s^2t : s^2t^{p+1}]$ for $p > 0$. Set $X = \overline{\varphi((k^\times)^2)} \subset \mathbb{P}^3$. Then $\gamma_X : X \dashrightarrow \mathbb{G}(2, \mathbb{P}^3) \simeq \mathbb{P}^3$ is locally described as

$$(s, t) \mapsto (0, 0, t^p) \in \mathbb{A}^3 \subset \mathbb{G}(2, \mathbb{P}^3).$$

Hence the fiber of γ_X over $(0,0,a) \in \mathbb{A}^3$ for $a \neq 0$ is

$$\varphi(k^\times \times \{a^{1/p}\}) = \left\{[1 : s : a^{1/p}s^2 : a^{p+1/p}s^2] \,|\, s \in k^\times\right\} \subset \mathbb{P}^3,$$

which is a conic.

By these results, we see that fibers of Gauss maps in $p > 0$ are quite different from those in $p = 0$. Hence we ask the following question.

Question 1.7. *What kind of difference appears about Gauss maps in $p > 0$ and $p = 0$?*

In [FI2], Furukawa and the author study Question 1.7 from the view point of images $\gamma_X(X)$, general fibers, and field extensions $K(X)/K(\gamma_X(X))$ induced by γ_X. In this article, we explain the results in [FI2].

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2. Main results

I.M. Landsberg and J. Piontkowski independently gave a necessary and sufficient condition for a subvariety $Y \subset \mathbb{G}(n, \mathbb{P}^N)$ to be the image of some n-dimensional variety $X \subset \mathbb{P}^N$ in $p = 0$. In other word, they characterized subvarieties of $\mathbb{G}(n, \mathbb{P}^N)$ which are written as the image of some Gauss map (see [FP, 2.4.7] and [IL, Theorem 3.4.8]). By their characterization, we know that in $p = 0$,

(1) not all subvariety of $\mathbb{G}(n, \mathbb{P}^N)$ is the image of some gauss maps. That is, there exists a subvariety $Y \subset \mathbb{G}(n, \mathbb{P}^N)$ with $\dim Y \leq n$ such that $Y \neq \overline{\gamma_X(X)}$ for any n-dimensional variety $X \subset \mathbb{P}^N$.
(2) If $Y = \overline{\gamma_X(X)}$ holds for some X, such X is uniquely determined by Y.

Furukawa [Fur] generalized their characterization to separable Gauss maps in any characteristics. In particular, (1), (2) hold under the assumption that γ_X is separable in $p \geq 0$.

On the other hand, Kaji show the following proposition about images of Gauss maps of curves in $p > 0$:

Proposition 2.1 ([Ka2, Appendix])**.** *Assume $p > 0$. For any curve $Y \subset \mathbb{G}(1, \mathbb{P}^N)$, there exists a curve $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$.*

More precisely, for any curve $Y \subset \mathbb{G}(1, \mathbb{P}^N)$ and any finite and inseparable field extension $L/K(Y)$, there exists a curve $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$, $K(X) = L$, and the field extension $K(X)/K(Y)$ induced by γ_X coincides with the given extension $L/K(Y)$.

Remark 2.2. For a fixed curve $Y \subset \mathbb{G}(1, \mathbb{P}^N)$, there exist infinitely many field extensions $L/K(Y)$ which are finite and inseparable. Thus there are infinitely many curves $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$ by Proposition 2.1. Hence the above (1), (2) does not hold in $p > 0$.

We generalize Proposition 2.1 to higher dimensional X, Y. In the rest of this article, a field means a finitely generated field over k. For a field extension L/K, we denote by $\delta_{L/K}$ the natural L-linear map $\Omega_{K/k} \otimes_K L \to \Omega_{L/k}$. The following is our main result.

Theorem 2.3. *Assume $p > 0$. For any variety $Y \subset \mathbb{G}(1, \mathbb{P}^N)$ with $1 \leq \dim Y \leq n$, there exists an n-dimensional variety $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$.*

More precisely, for any variety $Y \subset \mathbb{G}(1, \mathbb{P}^N)$ with $1 \leq \dim Y \leq n$ and any field extension $L/K(Y)$ such that $\mathrm{tr.deg}_k L = n$ and the linear map $\delta_{L/K(Y)}$ is zero, there exists an n-dimensional variety $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$, $K(X) = L$, and the field extension $K(X)/K(Y)$ induced by γ_X coincides with the given extension $L/K(Y)$.

Remark 2.4. We give some remarks about Theorem 2.3.

(a) For a field extension L/K, $\delta_{L/K}$ is injective if and only if L/K is separable. Hence the condition $\delta_{L/K} = 0$ is the "opposite" of the separability.

(b) If $\dim Y = 1$, $\delta_{L/K} : \Omega_{K/k} \otimes_K L \to \Omega_{L/k}$ is zero if and only if $\delta_{L/K}$ is not injective since $\dim_L \Omega_{K/k} \otimes_K L = 1$. Thus $\delta_{L/K} = 0$ if and only if L/K is inseparable in this case by (a). Hence Theorem 2.3 generalizes Proposition 2.1.

(c) If we replace the assumption $\delta_{L/K(Y)} = 0$ with "$L/K(Y)$ is inseparable" as in Proposition 2.1, Theorem 2.3 does not hold in general for $\dim Y \geq 2$, even if we assume that $L/K(Y)$ is finite. See [FI2, Example 3.4].

In Theorem 2.3, we consider images and field extensions. As a geometric version of Theorem 2.3, and as a generalization of Theorem 1.5 about general fibers of Gauss maps, we have the following theorem.

Theorem 2.5. *Assume $p > 0$. Let $Y \subset \mathbb{G}(n, \mathbb{P}^N)$ be a projective variety with $\dim Y \geq 1$ and let $\mathcal{F} \subset Y \times \mathbb{P}^{N'}$ be an n-dimensional projective variety such that the first projection $f : \mathcal{F} \to Y$ is surjective. Then there exist an n-dimensional projective variety $X \subset \mathbb{P}^N$ and a generically bijective rational map $h : X \dashrightarrow \mathcal{F}$ such that the Gauss map γ_X of X is equal to $f \circ h$. In particular, $\overline{\gamma_X(X)} = Y$ holds.*

Furthermore, if we assume $n \geq N'$, we can take $X \subset \mathbb{P}^N$ such that the fiber $\overline{\gamma_X^{-1}(y)}_{red} \subset \mathbb{P}^N$ of γ_X over general $y \in Y$ is projectively equivalent to

$F_y := f^{-1}(y)_{red} \subset \{y\} \times \mathbb{P}^{N'}$.

$$\begin{tikzcd}
X \arrow[r, dashed, "h", "gen.\ bij."'] \arrow[dr, dashed, "\gamma_X"'] & \mathcal{F} \subset Y \times \mathbb{P}^{N'} \arrow[d, "f"] \\
 & Y \subset \mathbb{G}(n, \mathbb{P}^N).
\end{tikzcd}$$

Roughly, Theorem 2.5 states that any surjective morphism $\mathcal{F} \to Y$ appears as a Gauss map up to generically bijective rational maps. We note that the assumption $n \geq N'$ is necessary in the last statement of Theorem 2.5 since for $y = [L] \in Y$, the general fiber $\overline{\gamma_X^{-1}(y)}_{red}$ must be contained in $L \simeq \mathbb{P}^n$.

3. Idea of proof

Let $X \subset \mathbb{P}^N$ be an n-dimensional projective variety. Then we have a rational map

$$(\gamma_X, \mathrm{id}_X) : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N,$$

where id_X is the inclusion of X into $\mathbb{P}^N$.

Definition 3.1. We define a subvariety $\Gamma_X \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N$ to be the closure of the image of $(\gamma_X, \mathrm{id}_X)$. In other word, Γ_X is the *graph* of the rational map γ_X.

Landsberg, Piontkowski, and Furukawa characterize images of separable Gauss maps. However, it seems that such characterization does not work well for inseparable Gauss maps.

On the other hand, $(\gamma_X, \mathrm{id}_X) : X \dashrightarrow \Gamma_X$ is birational since Γ_X is the graph of γ_X. In particular, $(\gamma_X, \mathrm{id}_X)$ is separable. Under this observation, we have the following idea:

Idea. *We characterize not images but graphs of Gauss maps to investigate not necessarily separable Gauss maps.*

To state a key theorem, we prepare some notation.

Since $\mathbb{G}(n, \mathbb{P}^N)$ is a Grassmannian, there exist locally free sheaves $\mathcal{S}$ and $\mathcal{Q}$ on $\mathbb{G}(n, \mathbb{P}^N)$ of ranks $N - n$ and $n + 1$ respectively with the exact sequence

$$0 \to \mathcal{S} \to H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(n,\mathbb{P}^N)} \to \mathcal{Q} \to 0.$$

Hence each point $y \in \mathbb{G}(n, \mathbb{P}^N)$ corresponds to the linear variety $\mathbb{P}(\mathcal{Q} \otimes k(y)) \subset \mathbb{P}^N$.

By the surjection $H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(n,\mathbb{P}^N)} \to \mathcal{Q}$, we have a projective bundle

$$\mathcal{U} := \mathbb{P}(\mathcal{Q}) \subset \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(n,\mathbb{P}^N)}) = \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N$$

over $\mathbb{G}(n, \mathbb{P}^N)$. Hence we have the tautological invertible sheaf $\mathcal{O}_{\mathcal{U}}(1)$ on $\mathcal{U}$ with the surjection $\mathrm{pr}_1^* \mathcal{Q} \twoheadrightarrow \mathcal{O}_{\mathcal{U}}(1)$. By definition, it holds that

$$\mathcal{U} = \big\{([L], x) \in \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N \,|\, x \in L\big\},$$

that is, $\mathcal{U}$ is the incidence variety.

For a variety $X' \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N$, let $\mathrm{pr}_1 : X' \to \mathbb{G}(n, \mathbb{P}^N)$ and $\mathrm{pr}_2 : X' \to \mathbb{P}^N$ be the first and second projections respectively. Since $\Omega_{\mathbb{G}(n,\mathbb{P}^N)} = \mathcal{Q}^\vee \otimes \mathcal{S}$ holds, there exists a natural homomorphism

$$\mathrm{pr}_1^*(\mathcal{Q}^\vee \otimes \mathcal{S}) = \mathrm{pr}_1^* \Omega_{\mathbb{G}(n,\mathbb{P}^N)} \to \Omega_{X'}$$

for a variety $X' \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N$. This homomorphism induces

$$\Phi : \mathrm{pr}_1^* \mathcal{Q}^\vee \to \Omega_{X'} \otimes \mathrm{pr}_1^* \mathcal{S}^\vee.$$

For an n-dimensional projective variety $X \subset \mathbb{P}^N$, the graph Γ_X is contained in $\mathcal{U}$ since $x \in \gamma_X(x) = \mathbb{T}_x X$ for a smooth point $x \in X$. Hence we have the following commutative diagram:

$$\begin{tikzcd}
& \Gamma_X \subset \mathcal{U} \subset \mathbb{G}(n,\mathbb{P}^N)\times\mathbb{P}^N \arrow[dl, bend right, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] \\
X \arrow[ur, dashed, "(\gamma_X,\mathrm{id}_X)"] \arrow[r, dashed, "\gamma_X"] & \mathbb{G}(n,\mathbb{P}^N).
\end{tikzcd}$$

The following theorem gives a characterization of graphs of Gauss maps. We note that this theorem holds for any characteristic.

Theorem 3.2. *Let $X' \subset \mathcal{U}$ be a subvariety of dimension n such that $\mathrm{pr}_2 : X' \to \mathbb{P}^N$ is generically finite and separable. Then the following are equivalent:*

(i) *$X' = \Gamma_X$ holds for some n-dimensional variety $X \subset \mathbb{P}^N$.*
(ii) *$X' = \Gamma_{X_0}$ holds for $X_0 = \mathrm{pr}_2(X')$.*
(iii) *The composite homomorphism $\mathcal{O}_{\mathcal{U}}(-1)|_{X'} \hookrightarrow \mathrm{pr}_1^* \mathcal{Q}^\vee \xrightarrow{\Phi} \Omega_{X'} \otimes \mathrm{pr}_1^* \mathcal{S}^\vee$ is the zero map at the generic point of X'.*

Proof. See [FI2]. □

Remark 3.3. By definition, Φ is the zero map at the generic point of X' if and only if so is $\mathrm{pr}_1^*(\mathcal{Q}^\vee \otimes \mathcal{S}) = \mathrm{pr}_1^* \Omega_{\mathbb{G}(n,\mathbb{P}^N)} \to \Omega_{X'}$, i.e. the $K(X)$-linear map $\delta_{K(X')/K(\mathrm{pr}_1(X'))}$ is zero. Hence (iii) in Theorem 3.2 holds automatically if $\delta_{K(X')/K(\mathrm{pr}_1(X'))} = 0$.

By using Theorem 3.2, we can show Theorem 2.3 as follows.

Sketch of proof of Theorem 2.3. Assume that there exists an n-dimensional variety $X' \subset \mathcal{U}$ which satisfies

- $\mathrm{pr}_2 : X' \to \mathbb{P}^N$ is generically finite and separable,
- $Y = \mathrm{pr}_1(X')$,
- the field extension $K(X')/K(Y)$ induced by pr_1 coincides with the given extension $L/K(Y)$.

Then X' is the graph of γ_X for $X = \mathrm{pr}_2(X') \subset \mathbb{P}^N$ by Theorem 3.2 and the assumption $\delta_{L/K(Y)} = 0$ (see Remark 3.3), and this X satisfies the conditions in Theorem 2.3.

Hence it suffices to find such X'. To find such X', it is enough to find $x_1, \ldots, x_n \in L$ such that

- $L/k(x_1, \ldots, x_n)$ is finite and separable,
- $L = K(Y)(x_1, \ldots, x_n)$.

In fact, we take X' to be the image of

$$\operatorname{Spec} L \xrightarrow{(f,g)} Y \times \mathbb{P}^n \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^n \overset{bir}{\sim} \mathbb{P}(\mathcal{Q}) = \mathcal{U},$$

where $f : \operatorname{Spec} L \to Y$ is induced by $L/K(Y)$ and $g : \operatorname{Spec} L \to \mathbb{P}^n$ is the morphism which is parametrized as $[1 : x_1 : \cdots : x_n]$.

By a suitable argument about the generators of field extensions, we can find such x_i and Theorem 2.3 is proved. $\square$

In Theorem 2.3, we consider only field extensions $L/K(Y)$ with $\delta_{L/K(Y)} = 0$. How about field extensions with $\delta_{L/K(Y)} \neq 0$? By using Theorem 3.2, we can show the following theorem about inseparable field extensions L/K, not necessarily $\delta_{L/K} = 0$.

Theorem 3.4. *Let L/K be an inseparable field extension. If $p \geq 3$ or $\operatorname{rank}_L \delta_{L/K}$ is even, there exists a hypersurface $X \subset \mathbb{P}^{n+1}$ for $n = \operatorname{tr.deg}_k L$ such that the extension $K(X)/K(\gamma_X(X))$ induced by γ_X coincides with the given extension L/K.*

The difference of Theorems 2.3 and 3.4 are as follows: In Theorem 2.3, we fix an embedding $\operatorname{Spec} K \hookrightarrow \mathbb{G}(n, \mathbb{P}^N)$ as $Y \subset \mathbb{G}(n, \mathbb{P}^N)$, and consider field extensions L/K such that $\delta_{L/K}$ is zero. In Theorem 3.4, we do not fix an embedding $\operatorname{Spec} K \hookrightarrow \mathbb{G}(n, \mathbb{P}^N)$, and consider L/K such that $\delta_{L/K}$ is not necessarily zero.

In the proof of Theorem 2.3, the condition (iii) in Theorem 3.2 automatically holds since $\Phi = 0$, which follows from the assumption $\delta_{L/K} = 0$. On the other hand, Φ is not zero in the setting of Theorem 3.4. Hence we need to take an embedding $\operatorname{Spec} K \hookrightarrow \mathbb{G}(n, \mathbb{P}^{n+1})$ and $x_1, \ldots, x_n \in L$ carefully so that the condition (iii) in Theorem 3.2 holds.

Remark 3.5. If $p = 2$, the behavior of Gauss maps is sometimes different from that in other characteristics. For example,

- For any hypersurface $X \subset \mathbb{P}^{n+1}$, $\operatorname{rank}_{K(X)} \delta_{K(X)/K(\gamma(X))}$ is even. Hence the statement of Theorem 3.4 does not hold if $p = 2$ and $\operatorname{rank}_L \delta_{L/K}$ is odd.
- For any variety $X \subset \mathbb{P}^N$ (which is not necessarily a hypersurface), $\operatorname{rank}_{K(X)} \delta_{K(X)/K(\gamma(X))}$ cannot be equal to one.

By Theorem 3.4 and Remark 3.5, we have the following question:

Question 3.6. *Assume $p = 2$. Let L/K be an inseparable field extension with $n = \operatorname{tr.deg}_k L$ and $\operatorname{rank}_L \delta_{L/K}$ is odd. Then is there an n-dimensional variety $X \subset \mathbb{P}^N$ with $N \geq n + 2$ such that $K(X)/K(\gamma(X)) = L/K$?*

Unfortunately, we do not know the answer of this question.

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